The relationship between some nonclassical Ramsey numbers

Tomasz Dzido
Institute of Informatics, University of Gdańsk
Wita Stwosza 57, 80-952 Gdańsk, Poland
tdz@inf.ug.edu.pl

and

Renata Zakrzewska
Mathematics Teaching and Distance Learning Centre
Gdańsk University of Technology
Al. Siedlicka 2, 80-233 Gdańsk, Poland
renata.zakrzewska@pg.gda.pl

January 31, 2017

Abstract

The upper (mixed) domination Ramsey number $u(m,n)(v(m,n))$ is the smallest integer p such that every 2-coloring of the edges of K_p with color red and blue, $\Gamma(B) \geq m$ or $\Gamma(R) \geq n$ ($\beta(R) \geq n$); where B and R is the subgraph of K_p induced by blue and red edges, respectively; $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of a graph G.

First, we prove that $v(3,n) = t(3,n)$ where $t(m,n)$ is the mixed irredundant Ramsey number i.e. the smallest integer p such that in every two-coloring (R,B) of the edges of K_p, $IR(B) \geq m$ or $\beta(R) \geq n$ ($IR(G)$ is the maximum cardinality of an irredundant set of G). To achieve this result we use a characterization of the upper domination perfect graphs in terms of forbidden induced subgraphs. By the equality we determine two previously unknown Ramsey numbers, namely $v(3,7) = 18$ and $v(3,8) = 22$.

In addition, we solve other four remaining open cases from Burger's *et. al.* article, which listed all nonclassical Ramsey numbers. We find that $u(3,7) = w(7,3) = 18$, $u(3,8) = w(8,3) = 21$, where $w(m,n)$ is the irredundant-domination Ramsey number introduced by Burger and Van Vuuren in 2011.

1 Introduction

For notation and graph theory terminology and definition we in general follow [4]. The *independence number of* G, denoted by $\beta(G)$, is the maximum cardinality among the independent sets of vertices of G. The *upper domination number of* G, denoted by $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G, while the *upper irredundance number of* G, denoted by $IR(G)$, is the maximum cardinality of an irredundant set of G.

Hence the parameters β, Γ, and IR are related by the following inequality chain, which was first noted by Cockayne and Hedetniemi [5] and has received considerable attention in the literature.

Theorem 1 ([5]) *For every graph G, $\beta(G) \leq \Gamma(G) \leq IR(G)$.*

Several hereditary classes of 'perfect' graphs have been defined and studied using the classical domination parameters. A graph G is called upper domination perfect (Γ-perfect) if $\beta(H) = \Gamma(H)$, for every induced subgraph H of G. In [6], Gutin and Zverovich gave a characterization of the upper domination perfect graphs in terms of forbidden induced subgraphs.

The notion of a Ramsey number may be defined in terms of independent sets in graphs. The classical Ramsey number $r(m,n)$ is the smallest integer p such that in every two-coloring (R,B) of the edges of K_p, $\beta(B) \geq m$ or $\beta(R) \geq n$. This definition has been generalised in many ways. Five generalisations that arose during the period 1989–2011 involve the so-called upper domination number and the irredundance number of a graph - parameters that are closely related to the independence number of a graph. The following five so-called nonclassical Ramsey numbers have previously been studied in the literature:

- *The irredundant Ramsey number* $s(m,n)$ is the smallest integer p such that in every two–coloring (R,B) of the edges of K_p, $IR(B) \geq m$ or $IR(R) \geq n$.

- *The irredundant–domination Ramsey number* $w(m,n)$ is the smallest integer p such that in every two–coloring (R,B) of the edges of K_p, $IR(B) \geq m$ or $\Gamma(R) \geq n$.
- *The mixed irredundant Ramsey number* $t(m,n)$ is the smallest integer p such that in every two–coloring (R,B) of the edges of K_p, $IR(B) \geq m$ or $\beta(R) \geq n$.
- *The upper domination Ramsey number* $u(m,n)$ is the smallest integer p such that in every two–coloring (R,B) of the edges of K_p, $\Gamma(B) \geq m$ or $\Gamma(R) \geq n$.
- *The mixed domination Ramsey number* $v(m,n)$ is the smallest integer p such that in every two–coloring (R,B) of the edges of K_p, $\Gamma(B) \geq m$ or $\beta(R) \geq n$.

It follows from Theorem 1 that for all m and n,

$$s(m,n) \leq w(m,n) \leq u(m,n) \leq v(m,n) \leq r(m,n)$$

and

$$s(m,n) \leq w(m,n) \leq t(m,n) \leq v(m,n) \leq r(m,n)$$

- it is possible that $u(m,n)$ and $t(m,n)$ are not related.

For any integer $q < v(m,n)$, a red-blue edge coloring (R,B) of the complete graph K_q for which $\Gamma(B) < m$ and $\beta(R) < n$ is called an avoidance $v(m,n,q)$-coloring. Furthermore, an avoidance $v(m,n,q)$-coloring is said to be extremal if $q = v(m,n) - 1$. (Extremal) avoidance $r(m,n,q)$, $t(m,n,q)$ and $u(m,n,q)$-colorings are defined similarly.

2 Known results

A nice compact summary of what is known was presented by Burger and van Vuuren in 2011 [2]. In this paper in table 5, known results on $s(m,n)$, $u(m,n)$, $t(m,n)$, $v(m,n)$ and $r(m,n)$ with appropriate extremal avoidance colorings are summarised. In 2014 Burger, Hattingh and Vuuren proved the next two values, namely $t(3,7) = 18$ and $t(3,8) = 22$ [3]. Finally, in 2016 Burger and van Vuuren established two last numbers, namely $s(3,8) = 21$ and $w(3,8) = 21$ [4].

We shall need the following result by Henning and Oellermann [7].

Lemma 2 ([7]) *Let G be a graph satisfying $\Gamma(G) \leq 2$ and $IR(G) = k$, where $k \geq 3$. Then $\overline{G}$ contains $K_{k+1,k+1} - M$ as an induced subgraph, where M is a matching of cardinality k.*

3 New results

3.1 $v(3, n) = t(3, n)$

First we prove the following general implication.

Theorem 3 *For all $h \geq 3$, $v(3, n) = t(3, n)$.*

Proof. From the inequality $v(m, n) \geq t(m, n)$, we have the lower bound $v(3, n) \geq t(3, n)$.

In order to establish the upper bound on $v(3, n)$, we show by contradiction that no avoidance $v(3, n, t(3, n))$-coloring exists. Suppose, to the contrary, that such a coloring indeed exists. For this coloring we have $\Gamma(B) \leq 2$ and $\beta(B) \leq n - 1$. This is followed by a value of $t(3, h)$ that in fact $IR(B) = x \geq 3$, $\Gamma(B) = \beta(B) = 2$. By Lemma 2 the red subgraph of such coloring contains $H = K_{x+1,x+1} - M$ as an induced subgraph, where M is a matching of cardinality x. H contains $H' = K_{3,3} - M'$, where M' is a matching of cardinality 3. To avoid a red triangle, two subgraphs induced by the vertices of the partite sets of the bipartite graph H' have only blue edges, so we obtain an induced red C_6. Now, we can use the characterization of the upper domination perfect graphs in terms of forbidden induced subgraphs - see Theorem 2.2 in [6]. Since the complement of an induced red C_6 is a graph G_1 from this Theorem, we obtain that the blue subgraph of an avoidance $v(3, n, t(3, n))$-coloring is not upper domination perfect and $\Gamma(B) > \beta(B) = 2$, a contradiction. □

An immediate consequence of Theorem 3 and $t(3, 7) = 18$, $t(3, 8) = 22$ [3] now follows.

Corollary 4 $v(3, 7) = 18$ *and* $v(3, 8) = 22$.

3.2 $u(3,7)=18$ and $u(3,8)=21$.

As an immediate consequence of Corollary 4, we have that $u(3,7) \leq 18$. On the other hand, we know that $s(3,7)=18$ (see ref. in [3]). Hence, we have the following result.

Corollary 5 $u(3,7)=18$.

By Corollary 4, $v(3,8)=22$. Since $s(3,8)=21$ [4] and $s(m,n) \leq u(m,n) \leq v(m,n)$, we obtain that $21 \leq u(3,8) \leq 22$. We have the following result which improves this inequality.

Theorem 6 $u(3,8)=21$.

Proof. We have the lower bound $u(3,8) \geq 21$. In order to establish the upper bound on $u(3,8)$, we show that no avoidance $u(3,8,21)$-coloring exists.

We know that $t(3,8)=22$ [3] and there are only three pairwise non-isomorphic extremal $t(3,8,21)$-colorings presented in Figure 11 in [4]. For these colorings, $IR(B) \leq 2$ and $\beta(R) \leq 7$. This is followed by a value of $t(3,7)=18$ [3] that in fact, for these colorings, $\beta(R)=7$. Now, we can use the characterization of the upper domination perfect graphs in terms of forbidden induced subgraphs - see Theorem 2.2 in [6]. Since all red graphs H_1, H_2, H_3 from Figure 11 in [4] contain a G_2 from this Theorem, we obtain that these red graphs are not upper domination perfect and $\Gamma(R) > \beta(B) = 7$, so these three colorings are not avoidance $u(3,8,21)$-colorings.

For the remaining edge 2-colorings of K_{21}, we have that $IR(B) \geq 3$ or $\Gamma(R) \geq \beta(R) \geq 8$. By Corollary 4, $v(3,7)=18$. Hence we need only consider the case $IR(B)=m \geq 3$, $\Gamma(B)=\beta(B)=2$ and $\beta(R)=7$. By Lemma 2 the red subgraph of such coloring contains $H = K_{m+1,m+1} - M$ as an induced subgraph, where M is a matching of cardinality m. H contains $H' = K_{3,3} - M'$, where M' is a matching of cardinality 3. To avoid a red triangle, two subgraphs induced by the vertices of the partite sets of the bipartite graph H' have only blue edges, so we obtain an induced red C_6. Now, we can use the characterization of the upper domination perfect graphs in terms of forbidden induced subgraphs - see Theorem 2.2 in [6]. Since the complement of an induced red C_6 is a graph G_1 from this Theorem, we obtain that the blue subgraph of these now considered colorings are not upper domination perfect and $\Gamma(B) > \beta(B) = 2$, a contradiction.

These all of the above considerations lead us to the conclusion that there is no an avoidance $u(3,8,21)$-coloring, and the proof is complete.

□

3.3 $w(7,3)=18$ and $w(8,3)=21$

Immediately from the definition of the irredundant Ramsey number $s(m,n)$ and the upper domination Ramsey number $u(m,n)$ we have:

Corollary 7 $s(7,3)=u(7,3)=18$.

and

Corollary 8 $s(8,3)=u(8,3)=21$.

As an immediate consequence of Corollary 7 and Corollary 8 and the inequality $s(m,n) \le w(m,n) \le u(m,n)$, we have the following result:

Corollary 9 $w(7,3)=18$ *and* $w(8,3)=21$.

Previous work by others (details and references can be found in [2, 4]), Theorem 6, Corollaries 4, 5, 9, give the values of nonclassical Ramsey numbers which are listed in the following table. We use bold style to denote the new bounds of this paper.

$s(m,n)$	$w(m,n)$	$u(m,n)$	$t(m,n)$	$v(m,n)$	$r(m,n)$
$s(3,3)=6$	$w(3,3)=6$	$u(3,3)=6$	$t(3,3)=6$	$v(3,3)=6$	$r(3,3)=6$
$s(3,4)=8$	$w(3,4)=8$	$u(3,4)=8$	$t(3,4)=9$	$v(3,4)=9$	$r(3,4)=9$
$s(4,3)=8$	$w(4,3)=6$	$u(4,3)=8$	$t(4,3)=8$	$v(4,3)=8$	$r(4,3)=9$
$s(3,5)=12$	$w(3,5)=12$	$u(3,5)=12$	$t(3,5)=12$	$v(3,5)=12$	$r(3,5)=14$
$s(5,3)=12$	$w(5,3)=12$	$u(5,3)=12$	$t(5,3)=13$	$v(5,3)=13$	$r(5,3)=14$
$s(3,6)=15$	$w(3,6)=15$	$u(3,6)=15$	$t(3,6)=15$	$v(3,6)=15$	$r(3,6)=18$
$s(6,3)=15$	$w(6,3)=15$	$u(6,3)=15$	$t(6,3)=15$	$v(6,3)=15$	$r(6,3)=18$
$s(3,7)=18$	$w(3,7)=18$	**u(3,7)=18**	$t(3,7)=18$	**v(3,7)=18**	$r(3,7)=23$
$s(7,3)=18$	**w(7,3)=18**	**u(7,3)=18**	$18 \le t(7,3) \le 23$	$18 \le v(7,3) \le 23$	$r(7,3)=23$
$s(3,8)=21$	$w(3,8)=21$	**u(3,8)=21**	$t(3,8)=22$	**v(3,8)=22**	$r(3,8)=28$
$s(8,3)=21$	**w(8,3)=21**	**u(8,3)=21**	$21 \le t(8,3) \le 28$	$21 \le v(8,3) \le 28$	$r(8,3)=28$
$s(4,4)=13$	$w(4,4)=13$	$u(4,4)=13$	$t(4,4)=14$	$v(4,4)=14$	$r(3,4)=18$

Table 1: All known values for nonclassical Ramsey numbers

4 Conclusion

In this paper we established the six new nonclassical Ramsey numbers. Using the properties of these numbers and the values in Table 1 it follows that the first open cases of $t(m, n)$ and $v(m, n)$ are now $t(7, 3)$ and $v(7, 3)$, respectively. In fact, there are the four smallest unknown Ramsey numbers involving the graph theoretic notion of domination, and are certainly worthy of further investigation.

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